3. Disguised Quadratics

Some of the most intimidating equations in the TMUA are quadratics in disguise. An expression built from exponentials, logarithms, trigonometric functions or higher powers can look like nothing you know how to solve. That is, until you notice a block that appears alongside its square.

Once you spot that, a single substitution, often in the form $u = f(x)$, can collapse the whole equation into a quadratic – something you can easily solve by factorising, completing the square or using the quadratic formula. The challenge is often how to use the solutions to this new quadratic in the context of the original question.

For example, if your substitution is $u = 3^x$, we will only permit answers where $u > 0$.

If your substitution was $u = 2^{\sin x}$, our values of u are stuck between 2^{-1} and 2^1, so we'd only accept answers where $\frac{1}{2} \le u \le 2$.

These questions are quite common on the TMUA, so have a go at spotting what the substitution is for yourself.

Example 1

Find the sum of the real solutions of the equation

$$4^x - 7 \cdot 2^x + 12 = 0$$

A 7

B 12

C $\log_2 7$

D $\log_2 12$

E $\log_2\left(\frac{4}{3}\right)$

We notice that $4^x = \left(2^x\right)^2$, so the equation is a quadratic in 2^x.

Substituting $u = 2^x$ gives $u^2 - 7u + 12 = 0$, which factorises as $(u-3)(u-4) = 0$.

So $u = 3$ or $u = 4$. Remember that $u = 2^x$, so $u > 0$. Both values are positive, so both are valid, and each gives exactly one solution: $2^x = 3$ gives $x = \log_2 3$, and $2^x = 4$ gives $x = 2$.

The sum of the solutions is $\log_2 3 + 2 = \log_2 3 + \log_2 4 = \log_2 12$, which is option D.

Example 2

How many solutions does the equation

$$2\sin^2\theta + \cos\theta + 1 = 0$$

have in the range $0 \le \theta \le 2\pi$?

A 0

B 1

C 2

D 3

E 4

The equation mixes $\sin^2\theta$ and $\cos\theta$, so we use $\sin^2\theta = 1 - \cos^2\theta$ to turn it into a quadratic in $\cos\theta$: $2(1 - \cos^2\theta) + \cos\theta + 1 = 0$.

This rearranges to $2\cos^2\theta - \cos\theta - 3 = 0$ and factorises as $(2\cos\theta - 3)(\cos\theta + 1) = 0$.

So $\cos\theta = 3/2$ or $\cos\theta = -1$.

The first must be rejected, since $\cos\theta$ can never exceed 1. Checking each root of the quadratic against the block's possible values is an essential step – don't forget it!

The second gives $\cos\theta = -1$, which in the range $0 \le \theta \le 2\pi$ holds only at $\theta = \pi$: the extreme value -1 is reached just once per cycle, not twice like a typical value.

There is exactly one solution, which is option B.

Example 3

Find the maximum value of

$$6 \cdot 2^{\sin x} - 4^{\sin x}$$

as x ranges over all real numbers.

A 11/4

B 8

C 9

D 12

E 16

F 27

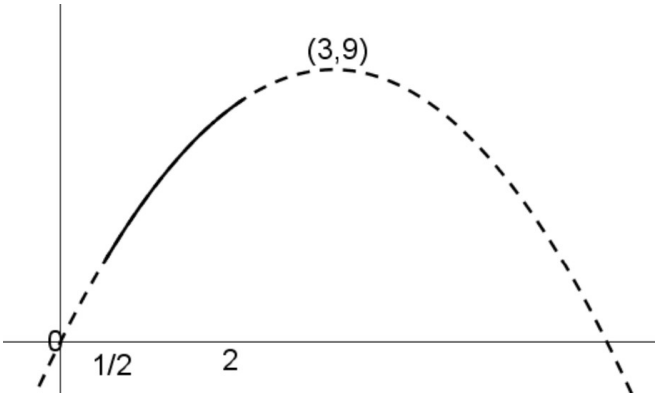
Since $4^{\sin x} = \left(2^{\sin x}\right)^2$, the expression is a quadratic in $u = 2^{\sin x}$.

The key observation is that this block does not range over all positive numbers.

Since $-1 \le \sin x \le 1$, we have $\frac{1}{2} \le u \le 2$, with every value in between reached at some point.

The expression is $f(u) = 6u - u^2 = 9 - (u-3)^2$, whose vertex sits at $u = 3$, outside the attainable interval, so the unconstrained maximum 9 is never reached.

It's useful to have a sketch like this in mind, or jotted down:



On the interval $\frac{1}{2} \le u \le 2$, which lies entirely to the left of the vertex, f is increasing, so its maximum is at the right-hand endpoint $u = 2$.

$f(2) = 12 - 4 = 8$, attained when $\sin x = 1$.

The maximum value is 8, which is option B.

Warm-up 1

Find the sum of all real solutions of the equation

$$9^x - 2 \cdot 3^{x+1} - 27 = 0$$

A 2

B 3

C 6

D 9

E 12

Warm-up 2

Find the minimum value of

$$\left(3^x - 5\right)\left(3^x + 1\right)$$

as x ranges over all real numbers.

A -21

B -9

C -5

D -1

E 0

F 2

G There is no minimum value.

Exercise 3

3.1

Find the sum of the real solutions of the equation

$$3^{2x+1} - 28 \cdot 3^x + 9 = 0$$

A -2

B 1

C 2

D 3

E $\frac{28}{3}$

3.2

Find the largest value of θ in the range $0 \leq \theta \leq 4\pi$ that satisfies

$$4\sin^2\theta + 4\cos\theta = 5$$

A $\frac{5\pi}{3}$

B $\frac{7\pi}{3}$

C $\frac{17\pi}{6}$

D $\frac{10\pi}{3}$

E $\frac{11\pi}{3}$

F $\frac{13\pi}{3}$

3.3

The minimum value of

$$4^x - 2^{x+a} + 12$$

as x ranges over all real numbers, is 3. Find the value of the real constant a.

A $\log_2 3$

B $\log_2\left(3\sqrt{2}\right)$

C $\log_2 6$

D $\log_2 9$

E $\log_2 18$

F $\log_2 36$

3.4

The equation

$$2^x - 11\left(\sqrt{2}\right)^x + 24 = 0$$

has exactly two real solutions, $x = p$ and $x = q$, where $p > q$. Find the value of $p - q$.

A 5

B $\log_2\left(\frac{9}{64}\right)$

C $\log_2\left(\frac{8}{3}\right)$

D $\log_2 5$

E $\log_2\left(\frac{64}{9}\right)$

F $\log_2 9$

G $\log_2\left(\frac{64}{3}\right)$

H $\log_2 576$

3.5

Find the sum of the real solutions of the equation

$$\left(x^2-4x\right)^2+7x^2-28x-60=0$$

A -12

B -7

C -5

D 0

E 4

F 5

G 8

3.6

How many solutions does the equation

$$2\cos^2\left(2x\right)+5\sin\left(2x\right)-4=0$$

have in the range $0^\circ \le x \le 240^\circ$, and what is the sum of these solutions?

A Number of solutions $=2$, Sum of solutions $=90^\circ$

B Number of solutions $=2$, Sum of solutions $=180^\circ$

C Number of solutions $=3$, Sum of solutions $=285^\circ$

D Number of solutions $=3$, Sum of solutions $=570^\circ$

E Number of solutions $=4$, Sum of solutions $=540^\circ$

F Number of solutions $=4$, Sum of solutions $=720^\circ$

3.7

Find the product of all real solutions of the equation

$$\left(\log_3 x\right)^4 - 5\left(\log_3\left(1/x\right)\right)^2 + 4 = 0$$

A The equation has no real solutions.

B $\frac{1}{243}$

C $\frac{1}{27}$

D 1

E 4

F 27

G 243

3.8

How many distinct real solutions does the equation

$$x\left(x-4\right) = 3\left|x-2\right| - 4$$

have?

A 0

B 1

C 2

D 3

E 4

3.9

The real numbers x and y satisfy the simultaneous equations

$$3^x + 3^y = 10$$

$$3^{x+y+1} = 72$$

Given that $x > y$, find the value of $x - y$.

A 2

B $\log_3\left(\frac{2}{3}\right)$

C $\log_3\left(\frac{3}{2}\right)$

D $\log_3 2$

E $\log_3 6$

F $\log_3 24$

3.10

Find the minimum value of

$$9^{\cos x} - 8 \cdot 3^{\cos x} + 10$$

as x ranges over all real numbers.

A -6

B -5

C 3

D $\frac{67}{9}$

E 10

F $\frac{115}{9}$

3.11

k is a real constant. Find the complete set of values of k for which the equation

$$4^x - k \cdot 2^x + 16 = 0$$

has exactly one real solution for x.

A $k = 8$ only

B $k = -8$ and $k = 8$ only

C $k > 8$

D $k \geq 8$

E $k < -8$ or $k > 8$

F $-8 < k < 8$

3.12

Find the sum of all real solutions of the equation

$$a^2 + \frac{1}{a^2} = 2\left(a - \frac{1}{a}\right) + 7$$

A The equation has no real solutions.

B -2

C $1 - \sqrt{6}$

D 0

E 1

F 2

G $1 + \sqrt{6}$

Challenge 1

How many distinct real solutions does the equation

$$2^{2x^2-4x+1} - 9 \cdot 2^{x^2-2x} + 4 = 0$$

have?

A 0

B 1

C 2

D 3

E 4

Challenge 2

k is a real constant. Find the complete set of values of k for which the equation

$$2\cos^2\theta - (2k+1)\cos\theta + k = 0$$

has exactly 4 distinct solutions in the range $0 \le \theta < 2\pi$.

A $-1 < k < 1$ and $k \neq 1/2$

B $-1 < k < 1$

C $-1 \le k \le 1$ and $k \neq 1/2$

D $-1 \le k \le 1$

E all real k with $k \neq 1/2$

F all real values of k

Hints

Warm-up 1 Both 9^x and 3^{x+1} can be rewritten in terms of 3^x, turning the equation into a quadratic in 3^x. Keep in mind which values 3^x can actually take.

Warm-up 2 The two brackets can be expressed in terms of 3^x. Multiply out and treat that block as a single variable, remembering which values it can take.

3.1 Write 3^{2x+1} in terms of 3^x, taking care with the extra factor of 3 that appears, and notice that the question asks for a sum of values of x, not of 3^x.

3.2 Use $\sin^2\theta+\cos^2\theta=1$ to get an equation in $\cos\theta$ alone, and don't stop your search at 2π, since the range goes further.

3.3 Write the expression as a quadratic in 2^x, then complete the square. Set the minimum value equal to 3 and find a. Take care when halving he coefficient of 2^x.

3.4 2^x is the square of $\left(\sqrt{2}\right)^x$, so the equation is a quadratic in $\left(\sqrt{2}\right)^x$. Take care at the end converting a value of $\left(\sqrt{2}\right)^x$ back into x.

3.5 Look at $7x^2-28x$ alongside the squared bracket – the whole equation is built from one repeated expression in x. After substituting and solving, check whether each value of that expression can actually be reached by a real x.

3.6 Replace $\cos^2(2x)$ using $\sin^2 2x+\cos^2 2x=1$ to get a quadratic in $\sin(2x)$. Then think carefully about the interval that $2x$ covers, and make sure the values you sum are values of x, not of $2x$.

3.7 First simplify $\log_3\left(\frac{1}{x}\right)$. The equation then involves only even powers of $\log_3 x$, and each valid value of $\left(\log_3 x\right)^2$ comes from two values of $\log_3 x$, rather than one.

3.8 Bring everything to one side and compare x^2-4x+4 with $|x-2|$. Once you have solved for $|x-2|$, count carefully how many x each value gives. It is not always two.

3.9 Let $a=3^x$ and $b=3^y$. One turns into $a+b$, while the other can be converted into a simple product using indices laws. Once you have the sum and product, form and solve a quadratic to find a and b.

3.10 Both terms are powers of $3^{\cos x}$, so a substitution converts this into a quadratic. Completed square form helps, but be careful to check whether the vertex of the quadratic is actually possible given the substitution.

3.11 With $u=2^x$, each solution for x corresponds to one positive root of a quadratic in u. Think about what the constant term 16 tells you about the signs of any real roots.

3.12 Use the identity $a^2+\frac{1}{a^2}=\left(a-\frac{1}{a}\right)^2+2$ to form a quadratic in $u=a-\frac{1}{a}$. Once you have found u, rearrange $u=a-\frac{1}{a}$ to get $a^2-ua-1=0$. For each value of u, use the coefficients to find the sum of the a-solutions without solving this quadratic.

Challenge 1 Compare the two exponents and you'll see one is built from the other. A single substitution will turn this into a quadratic.

Challenge 2 Treat this as a quadratic in $\cos\theta$. Work out its discriminant to help you factorise it, then count the solutions from each factor in the given interval. Check when either factor gives fewer than two solutions, and whether the same solutions have been counted twice.

Answers

Question	Answer
Warm-up 1	A
Warm-up 2	B
3.1	B
3.2	E
3.3	C
3.4	E
3.5	E
3.6	C
3.7	D
3.8	D
3.9	C
3.10	B
3.11	A
3.12	F
Challenge 1	D
Challenge 2	A

Model Solutions

Warm-up 1

We write everything in terms of 3^x: since $9^x = \left(3^x\right)^2$ and $3^{x+1} = 3 \cdot 3^x$, substituting $u = 3^x$ gives $u^2 - 6u - 27 = 0$, which factorises as $(u-9)(u+3) = 0$, so $u = 9$ or $u = -3$. Since 3^x is positive for every real x, $u = -3$ gives no solutions; $u = 9$ gives $x = 2$. The equation therefore has exactly one real solution, and the sum of all real solutions is 2, which is option A.

Warm-up 2

We substitute $u = 3^x$, which takes every positive value as x ranges over the reals. The expression becomes $(u-5)(u+1) = u^2 - 4u - 5 = (u-2)^2 - 9$. The vertex $u = 2$ is positive, so it is attained (at $x = \log_3 2$), and the minimum value is -9. (As u approaches 0 the expression approaches -5, but the genuine minimum sits at the vertex.) The answer is -9, which is option B.

3.1

Since $3^{2x+1} = 3\left(3^x\right)^2$, we substitute $u = 3^x$ to get $3u^2 - 28u + 9 = 0$, which factorises as $(3u-1)(u-9) = 0$, so $u = 1/3$ or $u = 9$. Both are positive, so both are valid: $3^x = 1/3$ gives $x = -1$, and $3^x = 9$ gives $x = 2$. The sum is $-1 + 2 = 1$. The answer is option B.

(Alternatively, the sum of the x-values is $\log_3$ of the product of the u-roots. If the x solutions are x_1 and x_2, we have $x_1 + x_2 = \log_3 u_1 + \log_3 u_2 = \log_3(u_1 u_2)$. Dividing the u quadratic by 3 gives $u^2 - \frac{28}{3}u + 3$, revealing that the product of the u-roots is 3, and $\log_3 3 = 1$ without solving for x. This technique is especially useful in cases where the solutions for x are hard or time consuming to find.)

3.2

We use $\sin^2\theta = 1 - \cos^2\theta$ to obtain $4 - 4\cos^2\theta + 4\cos\theta - 5 = 0$, that is $4\cos^2\theta - 4\cos\theta + 1 = 0$, which is the perfect square $(2\cos\theta - 1)^2 = 0$. The repeated root still gives just the one condition $\cos\theta = 1/2$. Over $0 \le \theta \le 4\pi$, two full revolutions, the solutions are $\pi/3$, $5\pi/3$, $7\pi/3$ and $11\pi/3$; note $13\pi/3$ exceeds $4\pi = 12\pi/3$ and is out of range. The largest is $11\pi/3$, which is option E.

3.3

For each fixed a the expression is a quadratic in $u = 2^x$: $f(u) = u^2 - 2^a u + 12$, where u ranges over all positive reals. Completing the square gives $\left(u - 2^{a-1}\right)^2 - \left(2^{a-1}\right)^2 + 12$.

The vertex is at $u = 2^{a-1}$, which is positive and therefore attainable for every a, and the minimum value is $12 - 2^{2a-2}$. The decisive equation is therefore $12 - 2^{2a-2} = 3$, so $2^{2a-2} = 9$, giving $2a - 2 = \log_2 9 = 2\log_2 3$ and $a = 1 + \log_2 3 = \log_2 6$.

The answer is option C

3.4

Both terms are powers of $u=\left(\sqrt{2}\right)^{x}=2^{x/2}$.

Since $2^{x}=u^{2}$, the equation becomes $u^{2}-11u+24=0$.

This factorises as $(u-3)(u-8)=0$, so $u=3$ or $u=8$. Both positive and so both valid.

Converting back: $2^{x/2}=8=2^{3}$ gives $x=\log_{2}64$ (which is 6, but let's leave like this for now, since we'll be subtracting logs later).

$2^{x/2}=3$ gives $\frac{x}{2}=\log_{2}3$, so $x=2\log_{2}3=\log_{2}9$.

$\log_{2}64$ is bigger than $\log_{2}9$, so $p-q=\log_{2}64-\log_{2}9=\log_{2}\left(\frac{64}{9}\right)$.

The answer is option E.

3.5

The key is to spot that $7x^{2}-28x=7\left(x^{2}-4x\right)$, so the whole equation is built from the single block $u=x^{2}-4x$.

This substitution gives $u^{2}+7u-60=0$, which factorises as $(u+12)(u-5)=0$.

So $u=5$ or $u=-12$. Each value of u opens its own quadratic in x.

For $u=5$: $x^{2}-4x-5=(x-5)(x+1)=0$, giving $x=5$ and $x=-1$.

For $u=-12$: $x^{2}-4x+12=0$ has discriminant $16-48<0$, so no real solutions.

The real solutions are 5 and -1, with sum 4, which is option E.

(If you know about roots of quartics from Vieta's formulas, do be careful. Expanding the full quartic and using the result $\alpha+\beta+\gamma+\delta=-\frac{b}{a}$ leads to an answer of 8, but this includes complex solutions, so isn't correct.)

3.6

Using $\cos^{2}(2x)=1-\sin^{2}(2x)$ gives $2\sin^{2}(2x)-5\sin(2x)+2=0$

Let $u=\sin 2x$, so we have $2u^{2}-5u+2=0$, so $(2u-1)(u-2)=0$, and $u=\frac{1}{2}$ or $u=2$.

So $\sin 2x=\frac{1}{2}$ or $\sin 2x=2$. Since $\sin(2x)=2$ is impossible, the only condition is $\sin(2x)=\frac{1}{2}$.

Since $0^o \le x \le 240^o$, we have $0^o \le 2x \le 480^o$.

Solving in this range gives $2x = 30^\circ$, 150° and 390°; the next candidate, 510°, exceeds 480° and must be excluded.

So $x = 15^\circ$, 75° and 195°: three solutions with sum 285°, which is option C.

3.7

Since $\log_3\left(\frac{1}{x}\right) = -\log_3 x$, its square equals $(\log_3 x)^2$, so with $L = \log_3 x$ the equation becomes $L^4 - 5L^2 + 4 = 0$: a quadratic in $u = L^2$.

Of course you can do a single substitution of $u = (\log_3 x)^2$. Whatever you prefer!

It factorises as $(u-1)(u-4) = 0$, so $u = 1$ or $u = 4$, so $L^2 = 1$ or $L^2 = 4$, giving the four values $L = 1, -1, 2, -2$ and hence $x = 3, \frac{1}{3}, 9, \frac{1}{9}$, all in the domain $x > 0$.

The product of these solutions is 1, which is option D.

(Notice that any time you have $\log x = \pm k$, the product of the two solutions, x^k and x^{-k}, will always be 1!)

3.8

Bringing everything to one side gives $x^2 - 4x + 4 - 3|x-2| = 0$, and the decisive observation is that $x^2 - 4x + 4 = (x-2)^2 = |x-2|^2$.

So with $u = |x-2|$, the equation is $u^2 - 3u = 0$, which is $u(u-3) = 0$.

This gives $u = 0$ or $u = 3$.

$|x-2| = 3$ means that $x - 2 = \pm 3$, which gives the two solutions $x = -1$ and $x = 5$.

While $|x-2| = 0$ gives only the single solution $x = 2$.

There are exactly 3 distinct real solutions, which is option D.

3.9

We set $a = 3^x$ and $b = 3^y$, both positive. The first equation says $a + b = 10$. For the second, $3^{x+y+1} = 3 \cdot 3^x \cdot 3^y = 3ab$, so $3ab = 72$ and $ab = 24$.

A sneaky move here is simply to spot that 4 and 6 are the only two numbers that add to 10 and multiply to 24. Since $x > y$ we should have $a > b$, so $a = 6$ and $b = 4$.

The slightly longer way that gives the same solution (of course) is to sub $b=\frac{24}{a}$ into $a+b=10$, leading to $a^2-10a+24=0$. This gives $a=4$ or $a=6$, and we then get $b=6$ or $b=4$.

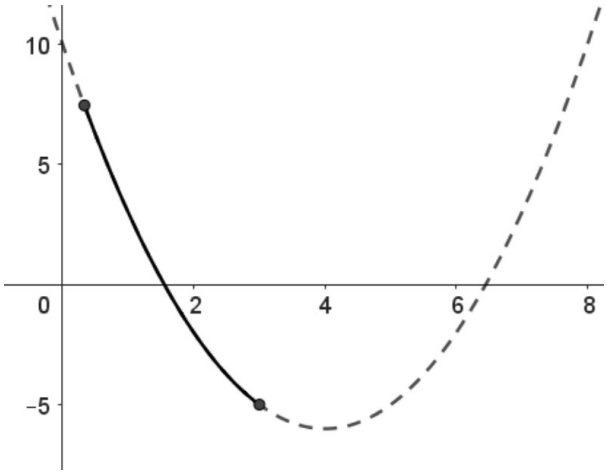
Since $x>y$, we have $a=6$, $b=4$.

Therefore $x-y=\log_3 6-\log_3 4=\log_3\left(\frac{6}{4}\right)=\log_3\left(\frac{3}{2}\right)$, which is option C.

3.10

Both terms are powers of $u=3^{\cos x}$, and since $-1\le\cos x\le 1$ this block ranges exactly over the interval $[1/3,3]$.

The expression is $f(u)=u^2-8u+10=(u-4)^2-6$, whose vertex $u=4$ lies outside the attainable interval, so the unconstrained minimum -6 is never reached. Here's the picture to have in mind or sketch out for yourself:



This quadratic in u is decreasing for all $u<4$, hence decreasing across the whole of $[1/3,3]$, so its minimum on the interval is at the right-hand endpoint $u=3$: $f(3)=9-24+10=-5$, attained when $\cos x=1$. The minimum value is -5, which is option B.

3.11

Substituting $u = 2^x$ gives us the quadratic $u^2 - ku + 16 = 0$. Any positive solution for u gives exactly one solution for x. The fancy terminology for this is a bijection from the real numbers (x) to the positive reals (u).

So we need this quadratic to have exactly one positive root. The decisive observation is that the product of the roots is $16 > 0$, so whenever the roots are real they have the same sign. One positive and one negative root is impossible.

The discriminant is $k^2 - 64$. If this is negative there are no real roots.

If $k^2 - 64 > 0$, there are two distinct roots. And they are either both positive (giving two solutions for x) or both negative (giving no real solutions for x).

That leaves the repeated-root cases: $k^2 - 64 = 0$. This has two solutions, $k = 8$ or $k = -8$.

$k = 8$ gives $(u - 4)^2 = 0$, so $u = 4$ and $x = 2$, exactly one solution.

$k = -8$ gives $u = -4$, which is impossible.

The complete set is $k = 8$ only, which is option A.

3.12

The decisive setup is the identity $a^2 + \frac{1}{a^2} = \left(a - \frac{1}{a}\right)^2 + 2$, with the plus sign because the cross-terms of the square are -2.

Substituting $u = a - \frac{1}{a}$ turns the equation into $u^2 + 2 = 2u + 7$, that is $u^2 - 2u - 5 = 0$.

The solutions to this are $u = 1 + \sqrt{6}$ and $u = 1 - \sqrt{6}$.

Let's not jump straight to subbing this into $u = a - \frac{1}{a}$. It would not be pretty!

So let's first think about where the u term ends up in the a quadratic.

Multiplying through by a leads to $a^2 - ua - 1 = 0$.

That central term $-ua$ tells us that the *sum* of the a roots is u. If this isn't obvious, just consider a simple quadratic like $x^2 - 4x + 3 = 0$ as a random example. The $-4x$ in middle tells us that the sum of the roots is 4. Check: $(x - 3)(x - 1) = 0$, $3 + 1 = 4$.

So we have two equations: $a^2 - (1 + \sqrt{6})a - 1 = 0$ and $a^2 - (1 - \sqrt{6})a - 1 = 0$.

The sum of the roots of the first equation is $1 + \sqrt{6}$, and for the second it's $1 - \sqrt{6}$.

Add these together for the overall sum of the four roots and we get 2.

The answer is option F.

Challenge 1

The two exponents are related: $2x^2 - 4x + 1 = 2\left(x^2 - 2x\right) + 1$, so with $u = 2^{x^2 - 2x}$ the first term is $2^{2\left(x^2 - 2x\right)} \cdot 2 = 2u^2$, and the equation becomes $2u^2 - 9u + 4 = 0$.

This factorises as $\left(2u - 1\right)\left(u - 4\right) = 0$, giving $u = \frac{1}{2}$ or $u = 4$.

Each value of u now opens a second quadratic through the inner expression: $x^2 - 2x = \log_2 u$.

For $u = 4$: $x^2 - 2x = 2$, so $\left(x - 1\right)^2 = 3$ and $x = 1 + \sqrt{3}$ or $x = 1 - \sqrt{3}$.

For $u = \frac{1}{2}$: $x^2 - 2x = -1$, so $\left(x - 1\right)^2 = 0$ and $x = 1$

In total there are 3 distinct real solutions, which is option D.

Challenge 2

We have a quadratic in $\cos\theta$, and the discriminant is $\left(2k + 1\right)^2 - 8k = 4k^2 - 4k + 1 = \left(2k - 1\right)^2$, a perfect square for every k.

The quadratic therefore factorises for all k as $\left(2\cos\theta - 1\right)\left(\cos\theta - k\right) = 0$, splitting the solutions into two families.

The first family, $\cos\theta = \frac{1}{2}$, contributes $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$ for every k.

The second family, $\cos\theta = k$, contributes two solutions when $-1 < k < 1$, exactly one when $k = 1$ or $k = -1$, none when $\left|k\right| > 1$, and when $k = 1/2$ its two solutions coincide with the first family's, adding nothing new.

Exactly 4 distinct solutions therefore requires the second family to contribute exactly two new solutions: $-1 < k < 1$ with $k \neq 1/2$, which is option A.